Vibration and Strength of Struts and Continuous Beams under End Thrusts.

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Summary.

On January 3, 1918, the authors presented a paper on "The Critical Loading of Struts and Structures." Since then the method of analysis has been extended to include the more general problem of the vibration of a continuous beam simply supported at any number of non-collinear points, the external loading being a constant longitudinal end thrust, varying from bay to bay and a periodic lateral loading. In addition, the supports are assumed in a state of periodic vibration. The flexural rigidity constant along each bay varies, however, from bay to bay. These conditions will then correspond closely with those originated in a wing spar of an aeroplane when in actual flight and influenced by engine throbbing. A very generalised form of the equation of three moments is derived, and the conditions for resonance and crippling expressed in a convenient determinantal form. The general equation where the restrictions are removed as regards constancy of end thrust and of flexural rigidity is then treated and illustrated in the particular case of the crippling of a strut of variable flexural rigidity.

Attention has at intervals been drawn to the effect of vibrations set up in aeroplane spars by external forces during flight, such as engine throbbing and wind forces. For very simple cases a few investigations exist to find under what conditions the results of this become serious on account of resonance, but no proper treatment of the actual problem and of the stresses brought into being appears to exist, especially in the case where the spar is under end thrust. In a previous communication* the static case of a beam supported at any number of non-collinear points and under the influence of lateral loading and end thrust varying from bay to bay was investigated, and the conditions for instability derived in a determinantal form. It will be seen that the previous discussion falls out as a mere particular case of the more general question developed in the present paper.

It is clear at the outset that the natural period of vibration and the conditions of resonance will be considerably affected by the presence of the end thrust, but it is not possible from simple considerations to anticipate to what extent. In any case, the problem is complicated in aerodynamic practice by the fact that the presence of bracing wires attached at intermediate points to the spar causes a variation in this thrust along the bay in the vertical plane and even in the longitudinal force plane. Under static conditions, it was previously understood that a structure of this type would fail through instability when the length of any one bay became equal to that of Euler's critical length, but it has already been proved in the authors' previous paper that this is not the case, and the real conditions under which failure arises were there stated. It appeared that the bays could not be regarded as isolated struts under end thrust, but that on account of interference effects between bays considerable modification arose. It is to be anticipated that in the analogous problem here to be investigated, corresponding interference due to vibration would result, and that likewise the spar could not be treated as a series of isolated struts under end thrust and external forced oscillation. It is proposed for the case of a spar supported simply at any number of points to investigate the effect of end thrust differing from bay to bay of the spar, upon the natural period of vibration, upon the conditions of resonance, and upon the distribution of bending moments and stresses in the structure. The ground will be cleared initially, and the manner of treatment exemplified by dealing in detail with the case of a beam under end thrust, simply supported at both ends, under a vertical periodic lateral loading, and a periodic bending moment at one end.

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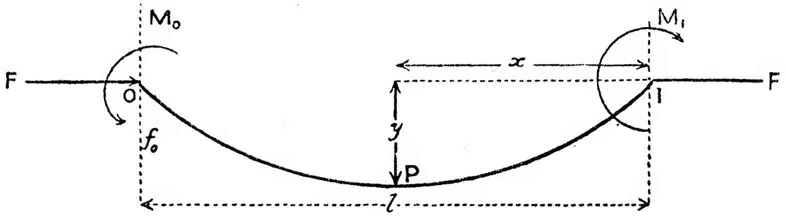
* 'Roy. Soc. Proc.,' A, vol. 94, p. 405 (1918).

Let the lateral loading be given by

$$w' = X \cos qt,$$

where X is a function of x only, and let the end thrust be F. If M_0 and M_1 be the bending moments at the two ends, and if P be any point in the beam



of co-ordinates (x, y) at any time t, then the moment on the portion to the left of P is

$$-M - M_0 + f_0(l-x) - Fy - \psi(x, t),$$

where f_0 is the shearing force at 0, and $\psi(x, t)$ is the moment of the lateral loading on that portion about the point P.

Since the portion of the beam is in motion under the influence of these moments, this must be equated to the rate of change of angular momentum of that portion, *i.e.*,

$$\int_{\xi=0}^{\xi=(l-x)} \frac{w}{g} \cdot \frac{\partial^2 y}{\partial t^2} \xi d\xi,$$

where w/g is the mass per unit length of the beam assumed constant along the spars and neglecting the rotation of the element. Equating these two expressions and differentiating twice with respect to x, we find

$$-\frac{\partial^2 \mathrm{M}}{\partial x^2} - \mathrm{F}\frac{\partial^2 y}{\partial x^2} - \frac{\partial^2 \psi}{\partial x^2} = \frac{w}{g}\frac{\partial^2 y}{\partial t^2}.$$

Remembering that $\mathrm{EI}\frac{\partial^2 y}{\partial x^2} = \mathrm{M}$ and that $\frac{\partial^2 \psi}{\partial x^2}$ is equal to the lateral loading, the equation reduces to

$$\frac{\partial^4 y}{\partial x^4} + \lambda^2 \frac{\partial^2 y}{\partial x^2} + \mu^2 \frac{\partial^2 y}{\partial t^2} = \mathrm{X} \cos qt, \tag{1}$$

where $\lambda^2 = \mathrm{F/EI}$ and $\mu^2 = w/g\mathrm{EI}$.

The problem will be solved by determining the solution of equation (1) which satisfies the conditions—

(*a*) $y = 0$ at $x = 0$ and $x = l$ for all time,

(*b*) $\mathrm{EI}\frac{\partial^2 y}{\partial x^2} = \mathrm{M}_0$ and M_1 at $x = 0$ and $x = l$ respectively, where M_0 and M_1 may be any given functions of t,

(*c*) $y =$ a known function of x for $t = 0$,

(*d*) $\partial y/\partial t =$ a known function of x for $t = 0$.

Since the equation has constant coefficients, solutions are of course additive.

Let the expression for y be supposed expressed in the form

$$y = \Sigma \mathrm{A}_n \cos q_n t + \Sigma \mathrm{B}_n \sin q_n t, \tag{2}$$

where A_n and B_n are functions of x, at present unknown, q_n unknown constants. Inserting (2) in equation (1) it follows that

$$\Sigma \cos q_n t \left\{ \frac{d^4 \mathrm{A}_n}{dx^4} + \lambda^2 \frac{d^2 \mathrm{A}_n}{dx^2} - \mu^2 q_n^2 \mathrm{A}_n \right\}$$

$$+ \Sigma \sin q_n t \left\{ \frac{d^4 \mathrm{B}_n}{dx^4} + \lambda^2 \frac{d^2 \mathrm{B}_n}{dx^2} - \mu^2 q_n^2 \mathrm{B}_n \right\}$$

$$= \mathrm{X} \cos qt. \tag{3}$$

It will be sufficient to equate the coefficients of cos q_nt and sin q_nt on each side of the equation, and we find accordingly

$$\left.\begin{aligned} \frac{d^4A_n}{dx^4}+\lambda^2\frac{d^2A_n}{dx^2}-\mu^2q_n^2A_n = 0, \\ \frac{d^4B_n}{dx^4}+\lambda^2\frac{d^2B_n}{dx^2}-\mu^2q_n^2B_n = 0, \end{aligned}\right\} \tag{4}$$

except where $q_n = q$, when

$$\frac{d^4A}{dx^4}+\lambda^2\frac{d^2A}{dx^2}-\mu^2q^2A = X. \tag{5}$$

To find a solution of these equations, the end conditions as restrictions on y and its derivatives must now be interpreted in terms of the restrictions on A and B, etc. Condition (a) which states that $y = 0$ at $x = 0$ and $x = l$ for all time requires generally that $A_n = 0$ and $B_n = 0$ at these points.

To illustrate the problem it will be sufficient to suppose

$$M_0 = 0,$$
$$M_1 = EIa\cos pt.$$

This corresponds with an applied periodic bending moment at one end, but since the solutions of the equations are additive, the presence of one single such term will not detract from the generality. Hence

$$\frac{\partial^2 y}{\partial x^2} = 0 \text{ at } x = l \text{ for all time,}$$

$$EI\frac{\partial^2 y}{\partial x^2} = EIa\cos pt \text{ at } x = 0 \text{ for all time.}$$

These interpreted for the A's and B's require that

$$d^2A_n/dx^2 = 0 \text{ at } x = l,$$
$$d^2B_n/dx^2 = 0 \text{ at } x = l \text{ and } x = 0,$$

and $$d^2A_n/dx^2 = 0 \text{ at } x = 0,$$

except when $q_n = p$, when

$$d^2A_p/dx^2 = a.$$

When $q_n \neq p$ the solution given by (4) is

$$A_n = P_n\sinh k_n'x+Q_n\cosh k_n'x+R_n\sin k_nx+S_n\cos k_nx,$$

where $\pm k_n'$, $\pm k_ni$ are roots of the equation

$$k^4+\lambda^2k^2-\mu^2q_n^2 = 0, \tag{6}$$

so that $$k_n^2+k_n'^2 = (\lambda^4+4\mu^2q_n^2)^{\frac{1}{2}}. \tag{7}$$

Inserting the boundary conditions

$$0 = Q_n + S_n,$$
$$0 = (P_n \sinh k_n' l + Q_n \cosh k_n' l) + (R_n \sin k_n l + S_n \cos k_n l),$$
$$0 = k_n'^2 Q_n - k_n^2 S_n,$$
$$0 = k_n'^2 (P_n \sinh k_n' l + Q_n \cosh k_n' l) - k_n^2 (R_n \sin k_n l + S_n \cos k_n l),$$

from which it follows that P_n, Q_n, R_n, and S_n are zero, unless

$$\sin k_n l = 0,$$

i.e., unless
$$k_n l = \pi,\ 2\pi,\ 3\pi,\ \ldots \text{ etc.,} \tag{8}$$

where P_n, Q_n, S_n are then zero and R_n is yet to be determined. This last condition may be written otherwise, viz.,

$$\frac{n^2\pi^2}{l^2} = \frac{\lambda^2 + \sqrt{(\lambda^4 + 4\mu^2 q_n^2)}}{2}$$

or
$$q_n^2 = \frac{n^2\pi^2 g \mathrm{EI}}{l^2 w}\left(\frac{n^2\pi^2}{l^2} - \frac{\mathrm{F}}{\mathrm{EI}}\right). \tag{9}$$

n is any integer, and as long as

$$\frac{\mathrm{F}l^2}{\mathrm{EI}} < n^2\pi^2$$

there is a real period.

For $n = 1$ when this becomes an equality, the normal first Euler's critical occurs, and above this value of the end thrust the period $2\pi/q_n$ becomes imaginary, so that the value of y increases exponentially, and the motion of the beam is unstable. The same argument applies for the higher criticals. It should be particularly remarked that the fact that the beam is in a state of vibration has not affected the value of the crippling load, but that the longitudinal thrust is such as to increase or decrease the period of natural vibration according as the force is compressive or tensive.

In the general expression for y there appear, therefore, only terms in $\cos q_n t$ and $\sin q_n t$ for which q_n has the value derived in equation (9), together with terms in $\cos pt$ and $\sin qt$ to be obtained as follows:

So far

$$y = \sum_{0}^{\infty} \sin k_n x\,(R_n \cos q_n t + R_n' \sin q_n t) + A \cos qt + A' \cos pt, \tag{10}$$

where q_n is given by (9).

The conditions that A and A′ are to satisfy are

(1) At $x = 0$ and $x = l$, A and A′ are zero.

(2) At $x = l$, $\dfrac{d^2A}{dx^2}$ and $\dfrac{d^2A'}{dx^2}$ are zero.

(3) At $x = 0$, $\dfrac{d^2A}{dx^2} = 0$ and $\dfrac{d^2A'}{dx^2} = a$,

where A and A′ satisfy the equations

$$\frac{d^4A}{dx^4}+\lambda^2\frac{d^2A}{dx^2}-\mu^2q^2A = X, \tag{11}$$

$$\frac{d^4A'}{dx^4}+\lambda^2\frac{d^2A'}{dx^2}-\mu^2p^2A' = 0. \tag{12}$$

Consider the expression for A′, viz.,

$$A' = P\sinh k'x+Q\cosh k'x+R\sin kx+S\cos kx,$$

where $\pm k'$ and $\pm ki$ are the roots of the equation

$$k^4+\lambda^2k^2-\mu^2p^2 = 0. \tag{13}$$

Inserting the end conditions, the following hold :—

$$0 = Q+S,$$
$$0 = (P\sinh k'l+Q\cosh k'l)+(R\sin kl+S\cos kl),$$
$$0 = k'^2(P\sinh k'l+Q\cosh k'l)-k^2(R\sin kl+S\cos kl).$$
$$a = k'^2Q-k^2S,$$

from which P, Q, R and S are easily determined in general as finite quantities. This fixes the term A′ cos pt in (10). Special circumstances arise, however, when

$$\sin kl = 0,$$

i.e., when $kl = \pi, 2\pi$, etc.

This implies that $2\pi/p$, the period of the externally applied bending moment at the end synchronises with the natural period of vibration for the whole beam under the end thrust. Under these circumstances P, Q, R and S become excessively large, and y also as a consequence of the synchronism.

To determine A we write

$$A = P_1\sinh k_1'x+Q_1\cosh k_1'x+R_1\sin k_1x+S_1\cos k_1x+\frac{X}{D^4+\lambda^2D^2-\mu^2q^2},$$

where $\pm k_1'$ and $\pm k_1i$ is obtained from

$$k^4+\lambda^2k^2-\mu^2q^2 = 0.$$

The end conditions are of course $A = 0$ and $d^2A/dx^2 = 0$ for $x = 0$ and $x = l$. Hence

$$0 = Q_1+S_1+Z(o), \text{ where } Z(x) = \frac{X}{D^2+\lambda^2D^2-\mu^2q^2},$$

$$0 = P_1\sinh k_1'l+Q_1\cosh k_1'l+R_1\sin k_1l+S_1\cos k_1l+Z(l),$$

$$0 = Q_1k_1'^2-S_1k_1^2+Z''(o),$$

$$0 = k_1'^2(P_1\sinh k_1'l+Q_1\cosh k_1'l)-k_1^2(R_1\sin k_1l+S_1\cos k_1l)+Z''(l).$$

From these four equations the values of the four constants P_1, Q_1, R_1, S_1 are easily derived as definite finite quantities, except in the special case when

$$k_1 l = \pi,\ 2\pi,\ \text{etc.},$$

i.e., when the lateral impressed force synchronises with a natural vibration of the beam under end thrusts alone. In this case the expression for y becomes excessively large.

The various terms in equation (10) are now specified except for the constants in the infinite series. These will be determined by expanding the given values of y and $\partial y/\partial t$ at $t = 0$ in a Fourier's series of period π/l. It appears from the above analysis that the vibrations which occur are three in type, viz., those corresponding with the natural period of vibration of the beam under end thrust with all the overtones, that corresponding with the forced lateral vibration, and that transmitted to it by the periodic bending moment at the end. It is to be anticipated on account of this last feature that in the case of a spar consisting of any number of bays, the vibrations in one bay will be transmitted to any other in the same manner as this periodic bending moment operates, and that consequently all the bays vibrate with the same periods.

Effect of Vibrations of Supports.—If in the previous analysis the end conditions that the two supports are supposed rigid, are modified to correspond with the case where these two supports execute vertical harmonic motions of the form

$$\delta_0 \cos ut \text{ and } \delta_1 \cos ut$$

respectively, then the analysis will proceed as before, except that condition (a) becomes $y = \delta_0 \cos ut$ and $\delta_1 \cos ut$ at $x = l$ and $x = 0$ respectively. From (2) by equating coefficients it will be seen that these imply that the term $\cos ut$ must arise in (2), and that, therefore, $A_u = \delta_0$ and δ_1 at $x = l$ and $x = 0$ respectively, the other values of A_n and B_n being zero at these points. The analysis then proceeds along the same lines and indicates that P_u, Q_u, R_u and S_u will be determined as definite finite constants unless the period of the vibration of the supports coincides with a free period of the beam under end thrust alone, in which case resonance takes place.

It appears, then, quite generally that resonance, and therefore excessive amplitude of vibration of the bay will occur when the period of any one of the externally applied forced oscillations become

$$\frac{2}{n} \cdot \sqrt{\left(\frac{l}{g}\right)} \sqrt{\left(\frac{wl^3}{\mathrm{EI}\,(n^2\pi^2 - \phi^2)}\right)},$$

where $\phi^2 = \mathrm{F}l^2/\mathrm{EI}$, and n is any integer.

Vibrations of a Wing Spar under End Thrust, varying from Bay to Bay due to Lift Wires and under Forced Loading due to Wind Forces and Engine Throbbing.—Regarding the spar as a separate entity in the aeroplane there are three principal types of force operating upon it which may set up and affect vibration. The longitudinal thrust, as already explained, is brought into being by the lift wires. In addition there are in the longitudinal force plane certain bracing wires connected to the spar at intermediate points which cause a variation to some extent of the force along each bay. For the moment, however, the effect of these longitudinal bracing wires will be ignored, and it will be assumed merely that the force along each bay is constant, but varies from bay to bay. In normal flight the throbbing of the engine sets up a periodic vibration in the structure, and this will be transmitted by the wires, struts, etc., to the spars. The variation in end thrust in each bay due to the effect of this on the wires will be neglected for the present, and it will be merely supposed that the result of the throbbing will be to impose a periodic displacement upon one or more supports of the spar. It is difficult to estimate up the relative importance of the effect of variation in wind forces such as are caused by gusts or oscillations of the machine, but it will be assumed that this is equivalent to a periodic lateral loading on each bay, and it will be quite sufficient to assume that the period of these forces is the same from bay to bay. It is proposed to investigate the effect of these forced oscillations upon the bending moments and stresses in the spars and upon the conditions under which instability and resonance arise.

Let (r) and $(r-1)$ be two neighbouring supports of a spar, and as before let P of co-ordinates (x, y) be any point in that bay at time t, then by the same method as already adopted the equation (of motion) of the rth bay is

$$\frac{\partial^4 y}{\partial x^4}+\lambda_r^2\frac{\partial^2 y}{\partial x^2}+\mu_r^2\frac{\partial^2 y}{\partial t^2}=w_r', \tag{14}$$

where

$$\lambda_r^2 = \mathrm{F}_r/\mathrm{E}_r\mathrm{I}_r \tag{15}$$

and

$$\mu_r^2 = w_r/g\mathrm{E}_r\mathrm{I}_r. \tag{16}$$

F_r = the longitudinal thrust in the rth bay,

$\mathrm{E}_r\mathrm{I}_r$ = the flexural rigidity,

w_r = weight per unit length assumed constant,

w_r' = lateral force per unit length = $\mathrm{X}\cos qt+\mathrm{X}_0$.

The equation (14) has constant coefficients, and therefore the solutions are additive. The boundary conditions that must be satisfied are as follows:—

$$(a) \text{ At } x = 0,\ y = d_r = \mathrm{D}_r \cos qt,$$

$$x = l_r,\ y = d_{r-1} = \mathrm{D}_{r-1} \cos qt,$$

where d is the deflection of a support measured from some datum line fixed in space.

$$(b) \text{ At } x = 0\ \mathrm{M} = \mathrm{M}_r,$$

$$x = l_r\ \mathrm{M} = \mathrm{M}_{r-1}.$$

These bending moments will be obtained by the appropriate moment equations to be given later.

$$(c) \text{ At } t = 0,\ y \text{ and } \partial y/\partial t \text{ are given functions of } x.$$

Generally, the forced vibrations may have frequencies other than those corresponding with cos qt, and, moreover, vibrations may occur which are not common to both the lateral loading and the vibrations of the supports. The above analysis, however, will include these cases if each frequency is taken separately, and the resulting moments, etc., added algebraically, since solutions of the differential equations are additive. When a vibration is not common, it could be treated as though present in both the lateral vibration and in the vibrations of the supports, except that one of the amplitudes is zero. Under these conditions, it will be perfectly general to consider only those problems in which every bay and every support receives forced vibrations of the same period but of different amplitudes.

Assume y of the form

$$y = \Sigma \mathrm{A}_n \cos q_n t + \Sigma \mathrm{B}_n \sin q_n t, \tag{17}$$

where A_n and B_n are functions of x alone, and the two summations may, for the present, be either finite or infinite series.

Inserting this in the equation of motion, we find

$$\Sigma\left(\frac{d^4\mathrm{A}_n}{dx^4} + \lambda_r{}^2\frac{d^2\mathrm{A}_n}{dx^2} - \mu_r{}^2 q_n{}^2 \mathrm{A}_n\right)\cos q_n t$$

$$+\Sigma\left(\frac{d^4\mathrm{B}_n}{dx^4} + \lambda_r{}^2\frac{d^2\mathrm{B}_n}{dx^2} - \mu_r{}^2 q_n{}^2 \mathrm{B}_n\right)\sin q_n t = \mathrm{X}\cos qt + \mathrm{X}_0.$$

Hence equating coefficients of cos $q_n t$ and sin $q_n t$ on both sides of this equation, it follows that

$$\frac{d^4\mathrm{B}_n}{dx^4} + \lambda_r{}^2\frac{d^2\mathrm{B}_n}{dx^2} - \mu_r{}^2 q_n{}^2 \mathrm{B}_n = 0 \tag{18}$$

for all values of n, and

$$\frac{d^4\mathrm{A}_n}{dx^4} + \lambda_r{}^2\frac{d^2\mathrm{A}_n}{dx^2} - \mu_r{}^2 q_n{}^2 \mathrm{A}_n = 0, \tag{19}$$

except when $q_n = q$, in which case

$$\frac{d^4A}{dx^4} + \lambda_r^2 \frac{d^2A}{dx^2} - \mu_r^2 qA = X, \tag{20}$$

and when $q_n = 0$, when

$$\frac{d^4A_0}{dx^4} + \lambda_r^2 \frac{d^2A_0}{dx^2} = X_0. \tag{20a}$$

Condition (a) for y will be satisfied if A_n and B_n are zero at $x = 0$ and $x = l_r$ for all values of q_n except $q_n = q$ when $A = D_r$ and D_{r-1} at $x = 0$ and $x = l_r$ respectively. For condition (b) we note that M_r, to be determined from the appropriate moment equation, must be expansible in the form

$$M_r = \Sigma r_n \cos qt + \Sigma r_n' \sin q_n t, \tag{21}$$

and differentiating (17) twice with respect to x, (b) reduces to

$$E_r I_r A_n'' = r_n, \text{ etc.}, \tag{22}$$

and

$$E_r I_r B_n'' = r_n', \text{ etc.} \tag{23}$$

Condition (c) merely states that ΣA_n and $\Sigma B_n q_n$ are given functions of x.

The solutions of equations (18) and (19) have already been given. They are

$$A_n = \alpha_n \sinh k_n' x + \beta_n \cosh k_n' x + \alpha_n' \sin k_n x + \beta_n' \cos k_n x, \tag{24}$$

where α_n, β_n, α_n', β_n' are obtained from the conditions that

$$A_n = 0 \text{ at } x = 0 \text{ and } x = l_r$$

and $\dfrac{d^2A_n}{dx^2} = \dfrac{r_n}{E_r I_r}$ and $\dfrac{(r+1)_n}{E_r I_r}$ at $x = 0$ and $x = l_r$ respectively,

while the solution of (20) is

$$A = \alpha \sinh k'x + \beta \cosh k'x + \alpha' \sin kx + \beta' \cos kx + \frac{X}{(D^4 + \lambda_r^2 D^2 - \mu_r^2 q)}, \tag{25}$$

where $\pm k_n'$ and $\pm k_n i$, and $\pm k'$ and $\pm ki$

are solutions of $k^4 + \lambda_r^2 k^2 - \mu_r^2 q_n^2 = 0$

and $k^4 + \lambda_r^2 k^2 - \mu_r^2 q^2 = 0$ respectively.

Confining attention for the moment to the case of $q_n = q$, we note that the boundary conditions demand

$$\text{At } x = 0, \; A = D_r \quad \text{and} \quad \frac{d^2A}{dx^2} = \frac{(r)}{E_r I_r},$$

$$x = l_r, \; A = D_{r-1} \text{ and } \frac{d^2A}{dx^2} = \frac{(r-1)}{E_r I_r}.$$

Inserting these conditions in (25), we obtain

$$\alpha = \frac{1}{(k^2+k'^2)\sinh k'l_r}\left[k^2\mathrm{D}_{r-1}+\frac{(r-1)}{\mathrm{E}_r\mathrm{I}_r}k^2\psi(l_r)-\psi''(l_r)-\beta\cosh k'l_r\right],$$

$$\beta = \frac{1}{k^2+k_1{}^2}\left[\frac{(r)}{\mathrm{E}_r\mathrm{I}_r}+k^2\mathrm{D}_r-k^2\psi(o)-\psi''(o)\right],$$

$$\alpha' = \frac{1}{(k^2+k'^2)\sin kl_r}\left[k'^2\mathrm{D}_{r-1}-\frac{(r-1)}{\mathrm{E}_r\mathrm{I}_r}-k'^2\psi(l_r)+\psi''(l_r)-\beta'\cos kl_r\right],$$

$$\beta' = \frac{1}{k^2+k'^2}\left[-\frac{(r)}{\mathrm{E}_r\mathrm{I}_r}+k'^2\mathrm{D}_r-k'^2\psi(o)+\psi''(o)\right],$$

where $$\psi(x) = \frac{\mathrm{X}}{\mathrm{D}^4+\lambda_r{}^2\mathrm{D}^2-\mu_r{}^2q^2}.$$

At $x = 0$, $$d\mathrm{A}/dx = k'\alpha+k\alpha'+\psi'(o)$$

$$= \frac{(r-1)}{l_r\mathrm{E}_r\mathrm{I}_r}\cdot\frac{1}{k^2+k'^2}\left\{\frac{k'l_r}{\sinh k'l_r}-\frac{kl_r}{\sin kl_r}\right\}$$

$$+\frac{(r)}{l_r\mathrm{E}_r\mathrm{I}_r}\cdot\frac{1}{k^2+k'^2}\left\{\frac{kl_r}{\tan kl_r}-\frac{k'l_r}{\tanh k'l_r}\right\}$$

$$+\mathrm{K}_r(\mathrm{F}, q), \text{ say,} \tag{26}$$

where K does not involve the bending moment, (r) and $(r-1)$. This may be written in the more concise form

$$(r-1)\mathfrak{A}_r(\mathrm{F}, q)+(r)\mathfrak{B}_r(\mathrm{F}, q)-\mathrm{K}_r(\mathrm{F}, q) = -\left(\frac{d\mathrm{A}}{dx}\right)_0, \tag{27}$$

where $$\mathfrak{A}_r(\mathrm{F}, q) = -\frac{l_r}{\mathrm{E}_r\mathrm{I}_r}\cdot\frac{1}{l_r{}^2(k^2+k'^2)}\left\{\frac{k'l_r}{\sinh k'l_r}-\frac{kl_r}{\sin kl_r}\right\}$$

$$= -\frac{l_r}{\mathrm{E}_r\mathrm{I}_r}\cdot\frac{1}{(\theta_r{}^2+\theta_r'^2)}\left\{\frac{\theta_r'}{\sinh\theta_r'}-\frac{\theta_r}{\sin\theta_r}\right\}$$

$$\mathfrak{B}_r(\mathrm{F}, q) = -\frac{l_r}{\mathrm{E}_r\mathrm{I}_r}\cdot\frac{1}{(\theta_r{}^2+\theta_r'^2)}\left\{\frac{\theta_r}{\tan\theta_r}-\frac{\theta_r'}{\tanh\theta_r'}\right\}.$$

The slope at the same point regarded as on the previous bay is

$$+(r)\mathfrak{B}_{r+1}(\mathrm{F}, q)+(r+1)\mathfrak{A}_{r+1}(\mathrm{F}, q)-\mathrm{K}'_{r+1}(\mathrm{F}, q) = \left(\frac{d\mathrm{A}}{dx}\right)_{lr+1}. \tag{28}$$

By equating these two expressions the extended form of the equations of three moments for a vibrating spar under an end thrust which varies from bay to bay is, writing m_r for (r),

$$\mathfrak{A}_r m_{r-1}+(\mathfrak{B}_r+\mathfrak{B}_{r+1})m_r+\mathfrak{A}_{r+1}m_{r+1} = +\mathrm{K}_r+\mathrm{K}'_{r+1}. \tag{29}$$

This extended form of the equation of three moments determines the coefficient of $\cos qt$ in the expansion (21) for the complete bending moment at any support. The remaining coefficients in the expansion for M are determined in a similar manner from equations (18) and (19). These coefficients, however, are brought into existence by the superposed natural vibration of the spar under end thrust, and if it be supposed, as is in practice quite legitimate, that slight damping is present, then they die out, and there are merely left coefficients due to the external oscillations. It is possible, of course, that the shape of the wing section might be such that the air forces brought into play as the wing vibrates would be of the nature to maintain the motion, and thus counterbalance and even over-balance any damping effect, but for the moment this state of affairs is not contemplated. The coefficients, $\mathfrak{A}_r$, $\mathfrak{B}_r$, etc., in the extended form of the equation of three moments are easily calculated for any bay from the roots of the biquadratic in k, while K_r and K'_{r+1} are known functions of the end thrust, lateral loading and q. If the end supports of a system occur at the extreme end of the spar so that there is no overhang, the bending moments at these points are zero, and by commencing at one end and considering successive pairs of bays the bending moments at all other supports are determined in exactly the same manner as by the ordinary method of three moments. If, however, the end supports are not actually at the end of the bays, but overhanging pieces occur, from a knowledge of the loading, etc., on these pieces a slight modification would require to be introduced.

In general, these equations will determine finite and definite bending moments at each support for a given period $2\pi/q$ of the external impressed force and of the motion of the supports, and for given values of F along each bay. When resonance occurs so that the amplitude of vibration becomes excessively large, the bending moments also will become excessive, and the resonance condition which determines the resonating period for the whole bay then takes the form

$$\begin{vmatrix} \mathfrak{B}_1+\mathfrak{B}_2, & \mathfrak{A}_2, & 0, & 0 & \ldots\ldots\ldots \\ \mathfrak{A}_2, & \mathfrak{B}_2+\mathfrak{B}_3, & \mathfrak{A}_3, & 0 & \ldots\ldots\ldots \\ 0 & \mathfrak{A}_3, & \ldots\ldots & \ldots\ldots & \ldots\ldots\ldots \\ & & & \ldots\ldots & 0 \\ & & & \ldots\ldots & \mathfrak{A}_{n-1} \\ 0, & 0, & 0 & \ldots\ldots & \mathfrak{B}_{n-1}+\mathfrak{B}_n \end{vmatrix} = 0. \qquad (30)$$

A reduction formula applicable perfectly generally to this determinant is clearly identical in form with that obtained in the previous communication, and need not be discussed here. As in the case of the stationary spar treated in that paper, a number of theorems arise corresponding with those dealing with clamped supports, etc.

Condition (30) gives the resonating period for any value of F, while for $q=0$ this condition is that required for crippling of the structure non-vibrating. This term corresponding to ($20a$) is always present, and therefore for all values of q the crippling load is the same as for the non-vibrating spar.

In cases where the effect of the end thrust on the spar is not great, it will be true to a first approximation to put $\lambda_r^2 = F_r/E_rI_r = 0$, in which event the terms in the previous discussion become considerably simplified.

The expressions for $\mathfrak{A}_r$ and $\mathfrak{B}_r$ now take the form

$$\mathfrak{A}_r = \frac{l_r}{E_rI_r} \cdot \frac{1}{\phi_r}\left(\frac{1}{\sin\phi_r} - \frac{1}{\sinh\phi_r}\right), \tag{31}$$

$$\mathfrak{B}_r = \frac{l_r}{E_rI_r} \cdot \frac{1}{\phi_r}\left(\frac{1}{\tanh\phi_r} - \frac{1}{\tan\phi_r}\right), \tag{32}$$

where

$$\phi_r = l_r\sqrt{(\mu_r q)} = l_r\left(q^2 \cdot \frac{w}{g\mathrm{EI}}\right)^{1/4}. \tag{33}$$

This formula is at once analogous to the results obtained in the earlier paper, for the case of a non-vibrating spar under end thrust where $\mathfrak{A}_r$ and $\mathfrak{B}_r$ are the sums of the corresponding tension and compression terms used in that communication,

i.e.,

$$\mathfrak{A}_r = \frac{l_r}{E_rI_r}(\alpha_r + \alpha_r'),$$

$$\mathfrak{B}_r = \frac{r}{E_rI_r}(\beta_r + \beta_r'),$$

ϕ_r now being of course $l_r\sqrt{(\mu_r q)}$.

As a simple illustration of these results, consider the case of two bays only of equal flexural rigidities and of lengths l_1 and l_2. If there be no end thrust the determinantal condition (30) reduces to

$$\mathfrak{B}_1 + \mathfrak{B}_2 = 0,$$

i.e.,

$$\frac{l_1}{\phi_1}\left(\frac{1}{\tanh\phi_1} - \frac{1}{\tan\phi_1}\right) + \frac{l^2}{\phi^2}\left(\frac{1}{\tanh\phi_2} - \frac{1}{\tan\phi_2}\right) = 0,$$

where

$$\frac{\phi_1}{l_1} = \left(q^2 \cdot \frac{w}{g\mathrm{EI}}\right)^{1/4} = \frac{\phi_2}{l_2},$$

$$\therefore \quad \frac{1}{\tanh\phi_1} + \frac{1}{\tanh\phi_2} = \frac{1}{\tan\phi_1} + \frac{1}{\tan\phi_2}.$$

For three bays of lengths l_1, l_2, l_1 the determinantal condition is

$$\begin{vmatrix} \mathfrak{B}_1 + \mathfrak{B}_2 & \mathfrak{A}_2 \\ \mathfrak{A}_2 & \mathfrak{B}_2 + \mathfrak{B}_1 \end{vmatrix} = 0,$$

or

$$(\mathfrak{B}_1 + \mathfrak{B}_2)^2 = \mathfrak{A}_2^2.$$

In the case where $l_1 \to 0$, *i.e.*, the ends of the bay are clamped,

$$\mathfrak{B}_1 \to 0,$$

and the condition becomes

$$\frac{1}{\phi_2^2}\left(\frac{1}{\tanh\phi_2}-\frac{1}{\tan\phi_2}\right)^2=\frac{1}{\phi_2^2}\left(\frac{1}{\sin\phi_2}-\frac{1}{\sinh\phi_2}\right)^2,$$

i.e.,
$$\cos\phi_2\cosh\phi_2 = 1,$$

i.e.,
$$\phi_2 = 3\pi/2 \text{ approximately},$$

i.e.,
$$q=\frac{9}{4}\cdot\frac{\pi^2}{l^2}\sqrt{\left(\frac{g\mathrm{EI}}{w}\right)},$$

which determines the resonating period.

For the case of a bar of length l_1, fixed at one end and supported at the other,

$$\mathfrak{B}_1+\mathfrak{B}_2 = 0,$$

where $l_2 \to 0$ and therefore $\mathfrak{B}_2 \to 0$.

Hence
$$\mathfrak{B}_1 = 0.$$

i.e.,
$$\frac{1}{\tanh\phi_1}-\frac{1}{\tan\phi_1} = 0,$$

giving as a first solution

$$\phi_1 = 3{\cdot}93 \text{ approx.},$$

i.e.,
$$q\ \ 1{\cdot}57\frac{\pi^2}{2}\sqrt{\left(\frac{g\mathrm{EI}}{w}\right)}.$$

The foregoing analysis has assumed that in each bay the end thrust, weight per unit length and flexural rigidity are constant. If this assumption be removed the general equation of motion can be obtained by an application of the same methods and expressed in the form

$$\frac{\partial^2}{\partial x^2}\left(\mathrm{EI}\frac{\partial^2 y}{\partial x^2}\right)+\frac{\partial}{\partial x}\left(\mathrm{F}\frac{\partial y}{\partial x}\right)+\frac{w}{g}\frac{\partial^2 y}{\partial t^2} = \mathrm{X}\cos qt,$$

where EI, F, and w are arbitrary functions of x.

A convenient general form may be found for the solution of this equation in a form particularly suitable for graphical integration, and at the same time the conditions for critical loading, etc., can be easily deduced. The method, although quite general, will be applied in the particular case of the crippling of a strut of variable flexural rigidity, but under constant end thrust.

The above equation then takes the form

$$\frac{d^2\mathrm{M}}{dx^2}+\frac{\mathrm{F}}{\mathrm{EI}}\mathrm{M} = 0, \qquad (34)$$

if there is no lateral load, since $\mathrm{M} = \mathrm{EI}\dfrac{d^2y}{dx^2}$.

It will be seen that the assumption that no lateral load exists is in reality no restriction as regards the present investigation. The condition for instability will be a relation that must be satisfied by F in order that the bending moments along the strut become infinite. This is in accordance with the point of view developed in the earlier communication.

If the strut be of length l, and pin jointed or simply supported at the ends $x = 0$ and $x = l$, then it can easily be verified that the solution of this equation subject to the external conditions due, say, to eccentric loading

$$M = M_0 \text{ at } x = 0, \qquad M = M_1 \text{ at } x = l,$$

is

$$M = S_0 - S_1 + S_2 - S_3 + \text{etc.}, \tag{35}$$

where

$$S_0 = A + A'x, \tag{36}$$

$$S_{n+1} = \int_0^x dx \int_0^x \frac{F}{EI} S_n dx, \tag{37}$$

where A and A′ are constants to be determined from the end conditions.

The series may be written in the following converging form evidently suitable for graphical integration :

$$\begin{aligned} M = A\left[1 - F\int_0^x dx \int_0^x \frac{dx}{EI} + F^2 \int_0^x dx \int_0^x \frac{dx}{EI} \int_0^x dx \int_0^x \frac{dx}{EI} - \text{etc.},\right. \\ + A'\left[x - F\int_0^x dx \int_0^x \frac{x dx}{EI} + F^2 \int_0^x dx \int_0^x \frac{dx}{EI} \int_0^x dx \int_0^x \frac{x dx}{EI} - \text{etc.},\right. \end{aligned} \tag{38}$$

where each term may be obtained from the preceding one by two simple graphical integrations.

This may be written

$$M = AU + A'V.$$

Inserting the end conditions

$$M_0 = AU_0 + A'V_0,$$

$$M_1 = AU_l + A'V_l,$$

but evidently $U_0 = 1$ and $V_0 = 0$ from the expressions for U and V. Hence

$$A = M_0,$$

$$A' = \frac{1}{V_l}(M_1 - M_0 U_l),$$

$$M = M_0 U + (M_1 - M_0 U_l)\frac{V}{V_l}.$$

The bending moment at any point is thus obtained in terms of two converging series whose terms alternate in sign. Upper and lower limits may, therefore, rapidly be set to the sums of these series.

For instability, M is to become infinite, and this can only take place when

$$V_l = 0, \tag{39}$$

unless

$$(M_1 - M_0 U_l) = 0. \tag{40}$$

The conditions for crippling, viz., $V_l = 0$, is thus

$$l - F\int_0^l dx \int_0^x \frac{x\,dx}{EI} + F^2 \int_0^l dx \int_0^x \frac{dx}{EI}\left\{\int_0^x dx \int_0^x \frac{x\,dx}{EI}\right\} - \text{etc.} \ldots = 0, \tag{41}$$

which is perfectly general for any variation in EI, and may be rapidly solved for F by graphical integration.

As a verification, suppose EI constant, then the series becomes

$$l - \frac{F}{EI}\frac{l^3}{3\,!} + \frac{F^2}{EI}\frac{l^5}{5\,!} - \frac{F^3}{EI}\frac{l^7}{7\,!} + \text{etc.} = 0,$$

or multiplying throughout by $\sqrt{(F/EI)}$ and writing

$$\phi = l\sqrt{(F/EI)},$$

$$\phi - \frac{\phi^3}{3\,!} + \frac{\phi^5}{5\,!} - \text{etc.} = 0, \tag{42}$$

i.e., $\sin\phi = 0$.

Since $F = 0$ is not a solution,

$$l\sqrt{(F/EI)} = \phi = \pi,\ 2\pi, \text{etc.},$$

the usual Eulerian result.

It is clear that the criterion used in the foregoing analysis for crippling, namely, that the bending moments become infinite, can be applied directly to any case where the solution of equation (34) is completely known. A typical case of this nature, well worth analysis in detail, is that afforded by the assumption that the variation in the square root of the flexural rigidity is parabolic, *e.g.*,

$$EI = EI_0\left(1 - \frac{x^2}{c^2}\right)^2 = \frac{EI_0}{c^4}(c^2 - x^2)^2 \tag{43}$$

At $x = l/2$,

$$I_l = I_0(1 - l^2/4c^2). \tag{44}$$

The equation for solution for constant end thrust is

$$\frac{d^2M}{dx^2} + \frac{F\,.\,c^4\,.\,M}{EI_0(c^2 - x^2)^2} = 0. \tag{45}$$

This equation is completely integrable, giving

$$M = (c^2 - x^2)^{1/2}\left[A\left(\frac{c+x}{c-x}\right)^{\lambda} + B\left(\frac{c-x}{c+x}\right)^{\lambda}\right], \tag{46}$$

where

$$\lambda = \sqrt{\left(\frac{Fc^2}{EI_0} - 1\right)}. \tag{47}$$

Inserting the end conditions,

$$\left.\begin{aligned} \mathrm{M} &= \mathrm{M}_1 \text{ at } l/2 \\ \mathrm{M} &= \mathrm{M}_2 \text{ at } l/2 \end{aligned}\right\} c > l/2,$$

the condition that M becomes infinite as before takes the form

$$\begin{vmatrix} \left(\dfrac{2c-l}{2c+l}\right)^{\lambda} & \left(\dfrac{2c+l}{2c-l}\right)^{\lambda} \\ \left(\dfrac{2c+l}{2c-l}\right)^{\lambda} & \left(\dfrac{2c-l}{2c+l}\right)^{\lambda} \end{vmatrix} = 0,$$

i.e.,

$$\left(\frac{2c+l}{2c-l}\right)^{2i\sqrt{\left(\frac{\mathrm{F}c^2}{\mathrm{EI}_0}-1\right)}} = 1,$$

or

$$e^{2i\sqrt{\left(\frac{\mathrm{F}c^2}{\mathrm{EI}_0}-1\right)}\log\left(\frac{2c+l}{2c-l}\right)} = 1. \tag{48}$$

Hence, solving for F,

$$\sqrt{\left(\frac{\mathrm{F}c^2}{\mathrm{EI}_0}-1\right)} \,.\, \log\frac{2c+l}{2c-l} = n\pi\,;$$

$$\therefore \qquad \frac{\mathrm{F}l^2}{\mathrm{EI}_0} = \left[1+\left\{\frac{n\pi}{\log\dfrac{2c+l}{2c-l}}\right\}^2\right] \cdot \frac{l^2}{c^2}. \tag{49}$$

Or putting $r = \sqrt{(\mathrm{I}_1/\mathrm{I}_0)} = 1-l^2/4c^2$,

$$\mathrm{F} = 4\mathrm{EI}_0\left(\frac{1-r}{l^2}\right)\left[1+\left\{\frac{n\pi}{\log\dfrac{1+\sqrt{(1-r)}}{1-\sqrt{(1-r)}}}\right\}^2\right]. \tag{50}$$

The limit of this expression when $r = 1$ corresponding with uniform section can easily be shown to give

$$\mathrm{F} = \frac{\mathrm{EI}_0}{l^2} \,.\, n^2\pi^2,$$

which is Euler's value.

The case when $n = 0$ must be excluded, for this would give

$$\mathrm{F}c^2 = \mathrm{EI}_0,$$

and the solution obtained is illegitimate, the real bending moments not becoming infinite. The first crippling load is thus

$$\mathrm{F} = \frac{4\mathrm{EI}}{l^2}(1-r)\left\{1+\frac{\pi^2}{\log\left(\dfrac{1+\sqrt{(1-r)}}{1-\sqrt{(1-r)}}\right)^2}\right\}, \tag{51}$$

where $r^2 = \mathrm{I}_1/\mathrm{I}_0$.

When $r = 1/2$, so that the moment of inertia at the ends is only $1/4$ that at the mid-section, this gives $\mathrm{F}l^2/\mathrm{EI}_0 = 8{\cdot}28$.

For a uniform strut whose moment of inertia is only 1/4 that at the mid-section of the above strut

$$\frac{\mathrm{F}l^2}{\mathrm{EI}_0} = \pi^2 = 9{\cdot}86.$$

Hence by tapering the strut to a moment of inertia at the tip equal to 1/4 that at the central section, the strength has been reduced by 16 per cent. The variation in strength does not appear to be greatly affected by moderate tapering. The last formula, therefore, should cover most practical cases.

Two-dimensional Solutions of Poisson's and Laplace's Equations.

Part I.—By Leonard Bairstow, F.R.S., C.B.E. Part II.—By Arthur Berry, M.A., Fellow of King's College, Cambridge.

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In the course of a preliminary investigation of fluid motion when viscosity is taken into account, it was found that a solution of Laplace's equation was obtained in very general form. The solution gives exact analytical expressions where the conditions of the problem admit of exact solution, but the chief claim to notice arises from the extremely general character of the method if graphical and mechanical integration are employed.

The problem has been approached from the engineering standpoint, and the absence of any proof of the convergency of the series developed will immediately be evident to the mathematician. The justification by trial in particular cases is all that is attempted at the present moment, but in those cases the result is wholly satisfactory. For the analytical solution of the problems I am indebted to Mr. Arthur Berry who has added a second part to this paper, and has generally acted as helpful critic.

For single closed boundaries of any shape, or for a combination of separate boundaries, the problem of finding a function which satisfies either Poisson's or Laplace's equation, admits of direct attack for the region inside a single boundary or between boundaries if there be more than one. For the region outside a single boundary there is a further condition to be satisfied infinitely far from the boundary.

The problems arising in calculating the torsion of solid or hollow cylinders, the flexure of beams, and the motion of an inviscid fluid round a cylinder